

HyperKähler Quotient Construction of BPS Monopole Moduli Spaces.

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Abstract

We use the HyperKähler quotient of flat space to obtain some monopole moduli space metrics in explicit form. Using this new description, we discuss their topology, completeness and isometries. We construct the moduli space metrics in the limit when some monopoles become massless, which corresponds to non-maximal symmetry breaking of the gauge group. We also introduce a new family of HyperKähler metrics which, depending on the “mass parameter” being positive or negative, give rise to either the asymptotic metric on the moduli space of many $SU(2)$ monopoles, or to previously unknown metrics. These new metrics are complete if one carries out the quotient of a non-zero level set of the moment map, but develop singularities when the zero-set is considered. These latter metrics are of relevance to the moduli spaces of vacua of three dimensional gauge theories for higher rank gauge groups. Finally, we make a few comments concerning the existence of closed or bound orbits on some of these manifolds and the integrability of the geodesic flow.

1 Introduction

The purpose of this paper is to provide a simple and explicit construction, using the HyperKähler quotient [1], of some HyperKähler metrics which have been used to check the S-duality hypothesis in $N = 4$ supersymmetric Yang-Mills theory in four space-time dimensions [2, 3, 4, 5] and which have also been recently applied to Yang-Mills theory in three space-time dimensions [6].

The use of the HyperKähler quotient construction in this context is not in itself new, but our treatment has the advantage that rather little machinery is necessary to obtain simple and tractable expressions for the metrics. Moreover, it allows us to analyse certain global properties of the manifolds, such as topology and completeness, with comparatively little effort. We are able to make some statements about the isometries and geodesics of these metrics which are not obvious from the explicit forms given in [2, 7, 8]. Another advantage of this approach is that it greatly simplifies the analysis of the metrics on monopole moduli spaces when some of the monopoles become massless. The construction permits an easy examination of singularities and how they may be resolved by changing the level sets of the moment map.

The plan of the paper is as follows. In section 2 we review the HyperKähler quotient construction and, in particular, how starting from a flat HyperKähler structure on $\mathbb{H}^{m+d} \equiv$

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$\mathbb{R}^{4(m+d)}$, we can obtain a $4m$ -dimensional HyperKähler metric, using a d -dimensional subgroup G of the Euclidean group $E(4m+4d)$. In all our examples there is a tri-holomorphic action of the torus group $T^m = U(1)^m$ on \mathbb{H}^{m+d} which commutes with the action of G and therefore induces a tri-holomorphic T^m action on the quotient. The general local forms of the metrics admitting a tri-holomorphic torus action have been written down previously by Lindström & Roček [9] and Pedersen & Poon [10], up to the solution of a set of linear partial differential equations. Further work on this type of metrics is contained in [11]. While in the most of the previous work on monopole moduli spaces the solutions were deduced from an asymptotic (Liénard-Wiechert) analysis of interactions of the monopoles, the quotient construction gives the required solutions directly. An advantage of this construction is that one can also imagine taking the limit of large m , which in some cases would correspond to many monopoles and in others to a gauge group $SU(m+2)$ for large m ; this can be done naturally in the HyperKähler quotient setting.

Section 3 is devoted to giving some explicit examples. We obtain the Lee-Weinberg-Yi metric on \mathbb{R}^{4m} [2] which is the relative moduli space of $m+1$ distinct fundamental monopoles with gauge group $SU(m+2)$ broken down to $U(1)^{m+1}$. As we shall show, the Lee-Weinberg-Yi metric is determined by m linearly independent vectors in \mathbb{R}^m whose matrix of inner products gives the reduced mass matrix of the monopoles. As well as the Lee-Weinberg-Yi metric, we use this technique to construct the Calabi metrics on $T^*(\mathbb{CP}^m)$, the Taubian-Calabi metrics on \mathbb{R}^{4m} and the cyclic ALE and ALF four-metrics. They are relevant for the limiting cases of zero and infinite monopole mass which are the subject of section 4. As the last and probably most elaborate example of section 3 we construct an apparently new class of metrics which include as a special case a positive mass parameter version of the asymptotic metric on the moduli space of $m+1$ identical $SU(2)$ monopoles [7]. The general metric is complete and depends on $\frac{1}{2}m(m-1)$ three-vectors ζ_{ab} . When all $\zeta_{ab} = 0$ the manifold has singularities for both positive and negative mass parameters. The topology of these metrics is rather complicated and we defer its consideration until a later publication. Metrics of this type have recently figured in studies of gauge theory in three dimensions [6].

In section 4 we discuss the limiting forms of the Lee-Weinberg-Yi metric when the masses of one or more monopoles either vanish or become infinite. In our formalism this is equivalent to one or more of the m vectors describing the metric becoming infinitely large or vanishing respectively, but one can still perform the quotient in these cases. If all but two monopoles become massless, and $SU(m+2)$ breaks to $SU(m) \times U(1)^2$, the metric on the relative moduli space is the Taubian-Calabi metric. It was obtained in [8] but not recognised as such. As we show, if $SU(m+2)$ is broken to $SU(m+2-k) \times U(1)^k$ the construction works just as well. At the end of this section we turn to the opposite limit when one or more monopoles become infinitely massive. If only one mass remains finite we obtain the $m-1$ centre ALF metric.

Finally in section 5 we discuss some aspects of the geodesic motion. In particular we use the quotient construction to show that neither the Lee-Weinberg-Yi nor the Taubian-Calabi metrics admit closed or even bound geodesics. We also make some remarks concerning the integrability of the geodesic flow. The last section contains a few concluding comments.

2 HyperKähler Quotients

In this section we shall recall some properties of the HyperKähler quotient construction [1] that we shall need and establish our notation. If $\{\mathcal{M}, g, I, J, K\}$ is a HyperKähler manifold and G a Lie group with Lie algebra \mathfrak{g} which acts on \mathcal{M} preserving the HyperKähler structure, there will

be an associated moment map $\mu : \mathcal{M} \rightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*$.

If we pick an element $\zeta \in \mathcal{Z}$, the centre of \mathfrak{g}^* , i.e. the invariant element of \mathfrak{g}^* under the co-adjoint action, then

$$X_\zeta := \mu^{-1}(\zeta)/G$$

is also a HyperKähler manifold. If G is compact and acts freely on $\mu^{-1}(\zeta)$ and \mathcal{M} is complete then X_ζ is also complete¹. If G does not act freely on $\mu^{-1}(\zeta)$ then X_ζ will, in general, have singularities.

The manifold \mathcal{M} may also admit another group K whose action preserves the HyperKähler structure on \mathcal{M} and commutes with the action of G . Then K will descend to X_ζ as a group of tri-holomorphic isometries.

Note that, at least locally, everything we have said about Riemannian metrics carries over in a straightforward way to metrics of signature $(4p, 4q)$. This comment will be of use later but, unless stated otherwise, we shall be concerned with the usual positive definite case.

In this paper we consider the case

$$\mathcal{M} = \mathbb{R}^{4n} \cong \mathbb{H}^n$$

with its standard flat metric. The group G is a (typically non-compact) subgroup of the Euclidean group $E(4n)$ which preserves the standard HyperKähler structure.

In order to establish notation we consider the case $n = 1$. We may identify points $(w, x, y, z) \in \mathbb{R}^4$ with a quaternion $q \in \mathbb{H}$:

$$q = w + ix + jy + kz. \quad (1)$$

The metric is

$$ds^2 = dq d\bar{q}, \quad (2)$$

where $\bar{q} = w - ix - jy - kz$ and the three Kähler forms are

$$-\frac{1}{2}dq \wedge d\bar{q} = i\omega_I + j\omega_J + k\omega_K. \quad (3)$$

The HyperKähler structure is invariant under real translations

$$q \rightarrow q + t, t \in \mathbb{R}, \quad (4)$$

and right multiplications by unit quaternions $\cong SU(2)$

$$q \rightarrow qp, \quad (5)$$

$p\bar{p} = 1$. By contrast left multiplication by a unit quaternion

$$q \rightarrow pq$$

is an isometry of the metric (2) but rotates the three Kähler forms. This may also be seen by noting that since the metric is flat we may identify the tangent space with \mathbb{H} , then the complex structures I, J, K act on \mathbb{H} by left multiplication by i, j, k . The action of I, J, K thus commutes with right multiplication.

¹In the examples of this paper \mathcal{M} and G are both non-compact, so to check the completeness of the quotient manifold one has to check both the freedom of G -action and the behaviour at infinity.

The moment map for real translations is

$$\mu = \frac{1}{2}(q - \bar{q}). \quad (6)$$

The one parameter family of right multiplications

$$q \rightarrow q e^{it}, \quad t \in (0, 2\pi] \quad (7)$$

has moment map²

$$\mu = \frac{1}{2}qi\bar{q}. \quad (8)$$

Away from the origin, $q \neq 0$, the $U(1)$ action (7) is free. The moment map (8) allows us to identify the orbit space with \mathbb{R}^3 , the origin corresponding to the fixed point set $q = 0$. The moment map (8) thus defines a Riemannian submersion $\mathbb{R}^4 \setminus 0 \rightarrow \mathbb{R}^3 \setminus 0$ whose fibres are circles S^1 .

For later purposes it will prove useful to express the flat metric (2) in coordinates adapted to the submersion. Any quaternion may be written as

$$q = ae^{i\psi/2}, \quad (9)$$

where the real coordinate $\psi \in (0, 4\pi]$ and a is pure imaginary, $a = -\bar{a}$. Then the $U(1)$ action is given by

$$\psi \rightarrow \psi + 2t. \quad (10)$$

The moment map (8) defines three cartesian coordinates \mathbf{r} by

$$\mathbf{r} = qi\bar{q} = ai\bar{a} = -aia. \quad (11)$$

A short calculation reveals that the flat metric (2) in coordinates (ψ, \mathbf{r}) becomes

$$ds^2 = \frac{1}{4} \left(\frac{1}{r} d\mathbf{r}^2 + r(d\psi + \boldsymbol{\omega} \cdot d\mathbf{r})^2 \right), \quad (12)$$

with $r = |\mathbf{r}|$ and

$$\text{curl } \boldsymbol{\omega} = \text{grad} \left(\frac{1}{r} \right),$$

and where the curl and grad operations are taken with respect to flat euclidean metric on \mathbb{R}^3 with cartesian coordinates \mathbf{r} . The metric (12) is singular at $r = 0 \equiv q = 0$ but this is merely a coordinate artefact arising from the $U(1)$ action (7) having a fixed point there. Away from the fixed point the metric (12) is defined on the standard Dirac circle bundle over $\mathbb{R}^3 \setminus 0$ and the horizontal one-form $(d\tau + \boldsymbol{\omega} \cdot d\mathbf{r})$ defines the standard Dirac monopole connection.

In the next section we shall use the form of the metric (12) for \mathbb{R}^4 and the Hyperkählerian $U(1)$ action (7) to construct various HyperKähler quotients of \mathbb{R}^{4n} .

²The awkward factor of 2 appears in the moment map so as to enable us to make further definitions more natural.

3 Explicit quotient constructions

In this section we will give examples of some HyperKähler metrics constructed from flat space. Before proceeding we note that all of our examples of $4m$ -dimensional HyperKähler metrics admit a tri-holomorphic T^m action which implies that locally the metrics may be cast in the form [10]:

$$ds^2 = \frac{1}{4}G_{ab}d\mathbf{r}_a \cdot d\mathbf{r}_b + \frac{1}{4}G^{ab}(d\tau_a + \boldsymbol{\omega}_{ac} \cdot d\mathbf{r}_c)(d\tau_b + \boldsymbol{\omega}_{bd} \cdot d\mathbf{r}_d), \quad (13)$$

where $a, b = 1 \dots m$, G^{ab} is the inverse of G_{ab} , and the Killing vector fields $\partial/\partial\tau_a$ generate the T^m action. Unless otherwise stated, we will assume Einstein summation convention.

The matrix G_{ab} and the one-form components $\boldsymbol{\omega}_{ab}$ satisfy certain *linear* equations which have the property that given a solution for the matrix G_{ab} one may determine the one forms up to gauge equivalence. Thus to identify a metric of this form we need only to calculate G_{ab} . We shall use this fact later to relate metrics obtained by the quotient construction to previously known forms. From the discussion in the previous section the reader should be able to recognise that if $r_a = |\mathbf{r}_a|$ and

$$G_{ab} = \frac{\delta_{ab}}{r_a} \quad (14)$$

then one obtains the flat metric on \mathbb{H}^m .

Because the torus action is tri-holomorphic it has an associated moment map which in the present case is given (up to a scalar multiple) by

$$\mu = \mathbf{r}_a, \quad (15)$$

which may be used to parameterise the space of orbits of the T^m action. Away from the degenerate orbits Δ of the torus action the map from X_ζ to \mathbb{R}^{3m} is therefore a Riemannian submersion. This defines a torus bundle over $\mathbb{R}^{3m} \setminus \Delta$ with a connection whose horizontal one-forms are just

$$A^a = G^{ab}(d\tau_a + \boldsymbol{\omega}_{ac} \cdot d\mathbf{r}_c). \quad (16)$$

Because HyperKähler metrics are necessarily Ricci flat, it follows from the Killing's equations that the exact 2-forms

$$F^a = dA^a \quad (17)$$

are co-closed

$$d^*F^a = 0, \quad (18)$$

and hence harmonic. By taking wedge products we obtain a useful supply of even-dimensional exact and co-closed harmonic forms. This construction yields the middle-dimensional Sen form for fundamental monopoles [5].

3.1 Taub-NUT space

The prototype case of the constructions we are interested in is that of Taub-NUT space. We will describe this in great detail and omit explicit calculations for the later cases. Choose

$$\mathcal{M} = \mathbb{H} \times \mathbb{H} \quad (19)$$

with quaternionic coordinates (q, w) . Let G be \mathbb{R} , $t \in \mathbb{R}$, with action

$$(q, w) \rightarrow (qe^{it}, w + \lambda t), \quad \lambda \in \mathbb{R}, \quad (20)$$

and moment map

$$\begin{aligned}\mu &= \frac{1}{2}qi\bar{q} + \frac{\lambda}{2}(w - \bar{w}) \\ &= \frac{1}{2}\mathbf{r} + \lambda\mathbf{y},\end{aligned}\tag{21}$$

where $w = (y + \mathbf{y}), y \in \mathbb{R}$. The flat metric on \mathcal{M} is

$$ds^2 = \frac{1}{4}\left(\frac{1}{r}d\mathbf{r}^2 + r(d\psi + \boldsymbol{\omega}.d\mathbf{r})^2\right) + dy^2 + d\mathbf{y}^2.\tag{22}$$

The action (20) corresponds to $(\psi, y) \rightarrow (\psi + 2t, y + \lambda t)$, which leaves $\tau = \psi - 2y/\lambda$ invariant. We set, without loss of generality,

$$\zeta = 0.\tag{23}$$

On the five-dimensional intersection of the three level sets $\mu^{-1}(0)$ one has $\mathbf{y} = -\mathbf{r}/2\lambda$ so the induced metric is:

$$ds^2 = \frac{1}{4}\left(\frac{1}{r}d\mathbf{r}^2 + r(d\tau + \frac{2}{\lambda}dy + \boldsymbol{\omega}.d\mathbf{r})^2\right) + dy^2 + \frac{1}{4\lambda^2}d\mathbf{r}^2.\tag{24}$$

The metric on $\mu^{-1}(0)/\mathbb{R}$ is obtained by projecting orthogonally to the Killing vector field $\partial/\partial y$. Completing the square in (24) gives

$$\begin{aligned}ds^2 &= \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)d\mathbf{r}^2 + \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)^{-1}\left(d\tau + \boldsymbol{\omega}.d\mathbf{r}\right)^2 \\ &\quad + \left(\frac{r}{\lambda^2} + 1\right)\left(dy + \frac{r\lambda}{2}\frac{(d\tau + \boldsymbol{\omega}.d\mathbf{r})}{(\frac{r}{\lambda^2} + 1)}\right)^2\end{aligned}$$

Thus the metric on $\mu^{-1}(0)/\mathbb{R}$ is

$$ds^2 = \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)d\mathbf{r}^2 + \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)^{-1}(d\tau + \boldsymbol{\omega}.d\mathbf{r})^2.\tag{25}$$

This is the standard form of the Taub-NUT metric with positive “mass parameter”. It becomes singular at $r = 0$ but since, as is easily seen, \mathbb{R} acts freely on $\mu^{-1}(0)$ this is a coordinate singularity. To obtain global coordinates on the quotient space note that on $\mu^{-1}(0)$, $\mathbf{y} = -qi\bar{q}/2$ and the \mathbb{R} action shifts y , so we may set $y = 0$. Thus q serves as a global coordinate and we can see that topologically the Taub-NUT metric is equivalent to \mathbb{R}^4 . Also if $\lambda \rightarrow \infty$ Taub-NUT metric (25) degenerates to a flat metric on \mathbb{R}^4 of the form (12).

The Taub-NUT metric with negative mass parameter may be obtained in an analogous way but now starting with the flat metric of signature (4, 4):

$$ds^2 = dq d\bar{q} - dw d\bar{w}.$$

Following the steps above yields the same metric (25) but with λ^2 replaced by $-\lambda^2$.

The \mathbb{R} action (20) commutes with the $U(1)$ action:

$$(q, w) \rightarrow (qe^{i\alpha}, w)$$

which descends to the Taub-NUT metric as the tri-holomorphic action $\tau \rightarrow \tau + 2\alpha$. In addition the action of the unit quaternions

$$(q, w) \rightarrow (pq, pw\bar{p}),$$

$p\bar{p} = 1$, the $SU(2)$ action, commutes with both of the previous actions and leaves $\mu^{-1}(0)$ invariant. Therefore the full isometry group of the Taub-NUT metric is $U(2)$.

3.2 The Lee-Weinberg-Yi metric

The simplest generalisation of Taub-NUT metric is perhaps the metric on the relative moduli space of distinct fundamental monopoles when the gauge group is $SU(m+2)$ broken down to its maximal torus $U(1)^{m+1}$ [2, 12]. The case $m = 1$ coincides with Taub-NUT metric which is the exact metric on the relative moduli space of $SU(3)$ fundamental monopoles [13, 3, 4]. We take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H}^m \quad (26)$$

with coordinates (q_a, w_a) , $a = 1, \dots, m$, and $G = \mathbb{R}^m = (t_1, \dots, t_m)$ with action

$$\begin{aligned} q_a &\rightarrow q_a e^{it_a} \quad (\text{no sum over } a), \\ w_a &\rightarrow w_a + \lambda_a^b t_b. \end{aligned} \quad (27)$$

The action of G commutes with the tri-holomorphic action of $K = T^m = U(1)^m = (\alpha_1, \dots, \alpha_m)$ given by

$$\begin{aligned} q_a &\rightarrow q_a e^{i\alpha_a} \quad (\text{no sum over } a), \\ w_a &\rightarrow w_a. \end{aligned} \quad (28)$$

The moment maps of the \mathbb{R}^m action are

$$\mu_a = \frac{1}{2} q_a i \bar{q}_a + \frac{1}{2} \lambda_a^b (w_b - \bar{w}_b), \quad (29)$$

where the $m \times m$ real matrix λ_a^b is taken to be non-singular. The Lee-Weinberg-Yi metric is then the induced metric on $\mu^{-1}(0)/\mathbb{R}^m$. The zero set of the moment maps is given by

$$\mu_a = \frac{1}{2} \mathbf{r}_a + \lambda_a^b \mathbf{y}_b = 0, \quad (30)$$

where $\mathbf{r}_a = q_a i \bar{q}_a$ and $\mathbf{y}_a = 1/2(w_a - \bar{w}_a)$. A short calculation shows that the metric on the quotient is

$$\begin{aligned} ds^2 &= \frac{1}{4} G_{ab} d\mathbf{r}_a \cdot d\mathbf{r}_b + \frac{1}{4} G^{ab} (d\tau_a + \boldsymbol{\omega}(\mathbf{r}_a) \cdot d\mathbf{r}_a) (d\tau_b + \boldsymbol{\omega}(\mathbf{r}_b) \cdot d\mathbf{r}_b), \\ G_{ab} &= \frac{\delta_{ab}}{r_a} + \mu_{ab}, \end{aligned} \quad (31)$$

where

$$\begin{aligned} \mu_{ab} &= (\nu^t \nu)_{ab} \\ \nu &\equiv \lambda^{-1}, \end{aligned} \quad (32)$$

and

$$\text{curl}_c \boldsymbol{\omega}(\mathbf{r}_a) = \text{grad}_c \left(\frac{1}{r_a} \right). \quad (33)$$

The tri-holomorphic action is generated by $\partial/\partial\tau_a$. Condition (30) is invariant under the action of the unit quaternions

$$\begin{aligned} q_a &\rightarrow p q_a, \\ w_a &\rightarrow p w_a \bar{p}. \end{aligned} \quad (34)$$

The metric on $\mu^{-1}(0)/\mathbb{R}^m$ is thus invariant under $SU(2)$ and a tri-holomorphic action of T^m corresponding to (28). As remarked above: to specify a $4m$ -dimensional HyperKähler metric with a tri-holomorphic T^m action it suffices to specify the $m \times m$ matrix G_{ab} as a function of the \mathbf{r}_a 's. The remaining parts of the metric may then be deduced directly. In the present case G_{ab} depends on the matrix μ_{ab} . Physically μ_{ab} is, up to a constant factor, the reduced mass matrix in the centre of mass frame of the monopoles. Geometrically it is related to the matrix of inner products of the m linearly independent translation vectors defining the \mathbb{R}^m action. The translation vectors $v^{(b)}, v \in \mathbb{R}^m, b = 1, \dots, m$ have components:

$$(v^{(b)})_a = \lambda_a^b.$$

Thus

$$g(v^{(a)}, v^{(b)}) = (\lambda^t \lambda)^{ab} = \mu^{ab}.$$

If one thinks of the vectors $v^{(b)}$ as defining a lattice Λ in \mathbb{R}^m with metric $g(v^{(a)}, v^{(b)})$ then μ_{ab} are the components of the metric on the reciprocal lattice Λ^* .

As long as the matrix λ_a^b is invertible we may use (30) to eliminate \mathbf{y}_b in favour of \mathbf{r}_a on $\mu^{-1}(0)$. It then follows, as it did for the Taub-NUT case, that the m quaternions q_a will serve as global coordinates for the Lee-Weinberg-Yi manifold which is therefore homeomorphic to \mathbb{R}^{4m} . The completeness of the metric follows from the fact that \mathbb{R}^m acts freely on $\mu^{-1}(0)$ and an examination of the metric near infinity.

If the matrix λ_a^b becomes singular or diverges, i.e. if the translation vectors cease to be linearly independent or become infinite, we are led to various degenerate cases associated with the reduced mass matrix μ_{ab} dropping in rank. Physically this is associated with enhanced symmetries due to appearance of massless monopoles. We shall discuss this in more detail in section 4.

Above we have considered monopoles of a specific gauge group $SU(m+2)$, in fact this construction may be easily generalised for any semi-simple group of rank $m+1$. In notation of [2], λ_a 's (not to be confused with the translation matrix λ_{ab}) are essentially inner products between the simple roots of the Dynkin diagram for the gauge group. One replaces the flat metric on \mathcal{M} by

$$\sum (\lambda_a dq_a d\bar{q}_a + dw_a d\bar{w}_a),$$

the Kähler forms by

$$-\frac{1}{2}\lambda_a dq_a \wedge d\bar{q}_a - \frac{1}{2}dw_a \wedge d\bar{w}_a,$$

and the action (27) by

$$\begin{aligned} q_a &\rightarrow q_a e^{it_a \lambda_a} \quad (\text{no sum}), \\ w_a &\rightarrow w_a + \lambda_a^b t_b. \end{aligned}$$

The form of the moment map (29) is unchanged.

3.3 The Calabi metrics on $T^*(\mathbb{CP}^m)$

The construction of the Calabi metric on $T^*(\mathbb{CP}^m)$ is perhaps the oldest of the HyperKähler constructions [14]. We describe it here because we shall need it later. We choose

$$\mathcal{M} = \mathbb{H}^{m+1} \tag{35}$$

with coordinates $q_a, a = 1, \dots, m+1$, and $G = U(1)$ with action

$$q_a \rightarrow q_a e^{it}, \quad t \in (0, 2\pi] \quad (36)$$

and moment map

$$\mu = \frac{1}{2} \sum q_a i \bar{q}_a = \frac{1}{2} \sum \mathbf{r}_a, \quad (37)$$

where $\mathbf{r}_a = q_a i \bar{q}_a$. The level sets of the moment map $\mu^{-1}(\zeta)$ are given by

$$\mu = \frac{1}{2} \sum \mathbf{r}_a = \zeta \quad (38)$$

where the 3-vector ζ must be non-vanishing if the action (36) is to be free. Let us make the following redefinition to make the formulas tidier:

$$\zeta = \frac{1}{2} \mathbf{x}.$$

Then the potential function G_{ij} in (13) is:

$$\begin{aligned} G_{ii} &= \frac{1}{|\mathbf{x} - \sum \mathbf{r}_i|} + \frac{1}{r_i} \\ G_{ij} &= \frac{1}{|\mathbf{x} - \sum \mathbf{r}_i|}, \quad i \neq j \end{aligned} \quad (39)$$

and $i, j = 1, \dots, m$. The action (36) commutes with tri-holomorphic action of $SU(m+1)$ given by

$$q_a \rightarrow q_a U_{ac}, \quad (40)$$

where U_{ac} is a $(m+1) \times (m+1)$ quaternion valued matrix with no j or k components satisfying

$$U_{ac} \bar{U}_{ab} = \delta_{cb},$$

$$\det U = 1.$$

Left multiplication by a unit quaternion

$$q_a \rightarrow p q_a$$

induces rotation of \mathbf{r}_a 's. If we choose p such that this is an $SO(2)$ rotation about the ζ direction it will leave $\mu^{-1}(\zeta)$ invariant. Such an $SO(2)$ action will preserve a single complex structure.

Thus the Calabi metric is invariant under the effective action of $U(m+1)/\mathbb{Z}_{m+1}$ of which $SU(m+1)/\mathbb{Z}_{m+1}$ acts tri-holomorphically. With respect to a privileged complex structure we have a holomorphic effective action of $U(m+1)/\mathbb{Z}_{m+1}$. The principal orbits are of the form $U(m+1)/U(m-1) \times U(1)$. There is a degenerate orbit of the form $U(m+1)/U(m) \times U(1) \cong \mathbb{CP}^m$ corresponding to the zero section of $T^*(\mathbb{CP}^m)$.

A recent theorem of Swann and Dancer [15] shows that the Calabi metric is the unique complete HyperKähler metric of dimension greater than four which is of cohomogeneity one³. If $\zeta = 0$ the metric becomes incomplete – it has an orbifold singularity at $q = 0$.

³i.e. a manifold on which the generic or principle orbit of the isometry group has real codimension one

3.4 The Taubian-Calabi metrics

The name Taubian-Calabi is due to [16]. Take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H} \quad (41)$$

with coordinates (q_a, w) , $a = 1, \dots, m$, and $G = \mathbb{R}$ with action

$$\begin{aligned} q_a &\rightarrow q_a e^{it}, \\ w &\rightarrow w + t, \end{aligned} \quad (42)$$

$t \in \mathbb{R}$. The moment map is

$$\mu = \frac{1}{2} \sum q_a i \bar{q}_a + \frac{(w - \bar{w})}{2}. \quad (43)$$

Without loss of generality $\mu^{-1}(0)$ is given by

$$\frac{1}{2} \sum \mathbf{r}_a + \mathbf{y} = 0, \quad (44)$$

where as before $\mathbf{r}_a = q_a i \bar{q}_a$ and $\mathbf{y} = 1/2(w - \bar{w})$. There is a T^m action on the \mathbb{H}^m factor commuting with G , therefore the metric is of the general form (13) with metric components G_{ab} :

$$\begin{aligned} G_{aa} &= 1 + \frac{1}{r_a} \\ G_{ab} &= 1, \quad a \neq b. \end{aligned} \quad (45)$$

X_0 has a tri-holomorphic right action of $U(m)/\mathbb{Z}_m$ and a left action of $SU(2)$. The total isometry group of the Taubian-Calabi metric is then $U(m) \times SU(2)$ up to a discrete factor. The principle orbits of $U(m)$ are $U(m)/U(m-2)$ which are $(4m-4)$ -dimensional. The left action of $SU(2)$ rotates the q_a 's and therefore \mathbf{r}_a 's but leaves invariant the phase of the q_a 's, thus it increases the dimension of a principle orbit by two. We conclude that the principle orbits of the Taubian-Calabi metric are of codimension two. As in the case of Lee-Weinberg-Yi metric, the q_a 's serve as global coordinates, and we get a complete metric on \mathbb{R}^{4m} . Setting $m = 1$ gives Taub-NUT metric. We will give a detailed discussion of the $m = 2$ case in the next section.

In addition to continuous symmetries the Taubian-Calabi metrics admit many discrete symmetries. There are m reflections $R_a : q_a \rightarrow -q_a$ and S_m permutation group on m letters, both acting tri-holomorphically. Their fixed point sets are totally geodesic and HyperKähler. In this way one sees that the $4m$ -dimensional Taubian-Calabi manifold contains as a totally geodesic submanifold the $4n$ -dimensional Taubian-Calabi manifold, for $n < m$.

3.5 The cyclic ALE metrics

These metrics constitute an example of gravitational multi-instantons, complete 4-dimensional solutions to vacuum Einstein equations, that were originally written down and discussed in [17]. We take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H} \quad (46)$$

with coordinates (q_a, q) , $a = 1, \dots, m$, and $G = T^m = (t_1, \dots, t_m)$ with action

$$\begin{aligned} q_a &\rightarrow q_a e^{it_a} \quad (\text{no sum}) \\ q &\rightarrow q e^{i \sum t_a}. \end{aligned} \quad (47)$$

The moment maps for this action are

$$\mu_a = \frac{1}{2}(q_a i \bar{q}_a + q i \bar{q}). \quad (48)$$

If $\mathbf{r}_a = q_a i \bar{q}_a$ and $\mathbf{r} = q i \bar{q}$, then $\mu^{-1}(\zeta)$ is given by

$$\frac{1}{2}\mathbf{r}_a = \zeta_a - \frac{1}{2}\mathbf{r}. \quad (49)$$

For future convenience define $\zeta_a = 1/2\mathbf{x}_a$, then the level sets of the moment map are:

$$\mathbf{r}_a = \mathbf{x}_a - \mathbf{r},$$

and the metric on X_ζ takes the multi-centre form

$$ds^2 = \frac{1}{4}V d\mathbf{r}^2 + \frac{1}{4}V^{-1}(d\tau + \boldsymbol{\omega} \cdot d\mathbf{r})^2, \quad (50)$$

with

$$V = \frac{1}{r} + \sum \frac{1}{|\mathbf{r} - \mathbf{x}_a|}, \quad (51)$$

and

$$\text{curl } \boldsymbol{\omega} = \text{grad } V. \quad (52)$$

Because

$$\mathbf{r}_a - \mathbf{r}_b = \mathbf{x}_a - \mathbf{x}_b, \quad (53)$$

we require $\zeta_a \neq \zeta_b$, $\forall a, b$ in order that the action of T^m be free. Note that the case $m = 1$ coincides with the Eguchi-Hanson metric on $T^*(\mathbb{CP}^1)$ which is the first of the Calabi series of metrics. The isometry group of the multi-instanton metrics is just $U(1)$ unless all the centres lie on a straight line in which case there is an extra $U(1)$ symmetry.

3.6 The cyclic ALF metrics

These are the so-called multi-Taub-NUT metrics constructed by Hawking in [18]. Take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H} \quad (54)$$

with coordinates (q_a, w) , $a = 1, \dots, m$, and $G = \mathbb{R}^m$ with moment map:

$$\mu_a = \frac{1}{2}\mathbf{r}_a + \mathbf{y}, \quad (55)$$

where $\mathbf{r}_a = q_a i \bar{q}_a$ and $\mathbf{y} = (w - \bar{w})/2$. As before make the following redefinitions:

$$\mathbf{y} = \frac{1}{2}\mathbf{r}, \quad \mathbf{x}_a = \frac{1}{2}\zeta_a.$$

The metric on X_ζ is again of multi-centre form (50) but this time with

$$V = 1 + \sum \frac{1}{|\mathbf{r} - \mathbf{x}_a|}. \quad (56)$$

We must require $\zeta_a \neq \zeta_b$ to avoid orbifold singularities at the coincidence points, that is when two or more centres coincide. The ordinary Taub-NUT metric is the $m = 1$ case.

3.7 The asymptotic metric on the moduli space of $SU(2)$ monopoles and its variations

A more complicated example which also generalises Taub-NUT space corresponds to a form of the asymptotic metric on the moduli space of m $SU(2)$ BPS monopoles [7]. We begin by constructing the analogues of the Taub-NUT metric with positive mass parameter and then go on to consider the analogue of the Taub-NUT metric with negative mass parameter. It is this latter case which applies to the behaviour of $SU(2)$ BPS monopoles at large separation.

We choose

$$\mathcal{M} = \mathbb{H}^{\frac{1}{2}m(m-1)} \times \mathbb{H}^m \quad (57)$$

with coordinates (q_{ab}, w_a) , $a = 1, \dots, m$, $a < b$. The group G is taken to be $\mathbb{R}^{\frac{1}{2}m(m-1)} = (t_{ab})$ with action

$$\begin{aligned} q_{ab} &\rightarrow q_{ab} e^{it_{ab}}, \\ w_a &\rightarrow w_a + \sum_c t_{ac}, \end{aligned} \quad (58)$$

where $t_{ac} = -t_{ca}$ for $c < a$. The moment maps are

$$\mu_{ab} = \frac{1}{2} \mathbf{r}_{ab} - (\mathbf{y}_a - \mathbf{y}_b), \quad (59)$$

where $\mathbf{r}_{ab} = q_{ab} i \bar{q}_{ab}$ and $\mathbf{y}_a = (w_a - \bar{w}_a)/2$, then $\mu^{-1}(\zeta)$ is given by

$$\frac{1}{2} \mathbf{r}_{ab} = \mathbf{y}_a - \mathbf{y}_b + \zeta_{ab}. \quad (60)$$

Using (60) one may eliminate the \mathbf{r}_{ab} 's in favour of the \mathbf{r}_a 's, the quotient constricts eliminates the $\frac{1}{2}m(m-1)$ phases of the q_{ab} 's so one use the m quaternions w_a as local coordinates on X_ζ . Make the following redefinitions:

$$\mathbf{r}_a = \frac{1}{2} \mathbf{y}_a, \quad \mathbf{x}_{ab} = \frac{1}{2} \zeta_{ab}.$$

In these coordinates the metric is of the form (13) with potential functions given by:

$$\begin{aligned} G_{aa} &= 1 + \sum_{b \neq a} \frac{1}{|\mathbf{r}_a - \mathbf{r}_b + \mathbf{x}_{ab}|} \quad (\text{no sum over } a), \\ G_{ab} &= -\frac{1}{|\mathbf{r}_a - \mathbf{r}_b + \mathbf{x}_{ab}|}. \end{aligned} \quad (61)$$

The metric constructed by Gibbons and Manton in [7] is the “negative mass” version to obtain which one must take the flat metric on \mathcal{M} to be:

$$ds^2 = dq_{ab} d\bar{q}_{ab} - dw_a d\bar{w}_a \quad (62)$$

and choose $\zeta_{ab} = 0$. As pointed out in [7] the case of positive mass parameter appears to be relevant to the motion of $a = 1$ black holes. It also arises in three-dimensional gauge theory. Physically the coordinates w_a correspond to the positions and internal phases of the monopoles. Just as in the case of Lee-Weinberg-Yi more complicated metrics may be constructed by introducing weights.

The construction is invariant under m real translations of the w_a 's:

$$q_{ab} \rightarrow q_{ab} \tag{63}$$

$$w_a \rightarrow w_a + t_a \tag{64}$$

The global behaviour of these metrics is quite complicated, despite the simplicity of the construction, and we hope to return to them in a future publication. Note, as in the case of the Taubian-Calabi metrics, the metric with ζ_{ab} admit various discrete symmetries, e.g. reflections and permutation groups, as tri-holomorphic isometries. It follows that the $4m$ -dimensional metric contains totally geodesic copies of the first non-trivial case $m = 2$. This is presumably related to the observation of Bielawski that the exact $SU(2)$ moduli space of m monopoles always admits a totally geodesic copy of the Atiyah-Hitchin manifold [19].

4 Zero- and Infinite-Mass Monopoles

4.1 Massless Monopoles

In section 3.2 we showed how using HyperKähler quotient construction one may obtain the exact metric on the relative moduli space of fundamental $SU(m+2)$ monopoles when the gauge group is maximally broken. It is interesting to ask how the metric will change if some of the monopoles become massless, that is when the broken gauge group contains a non-abelian factor. This question was addressed in a recent paper of Lee, Weinberg and Yi [4]. The argument used allowed the authors to obtain the metric for the case

$$SU(m+2) \rightarrow SU(m) \times U(1)^2,$$

but did not yield an explicit answer for the more general case

$$SU(m+2) \rightarrow SU(m+2-k) \times U(1)^k, \tag{65}$$

$k = 2, \dots, m+1$. Using the HyperKähler quotient method, however, greatly simplifies the task and we construct these metrics below. We will also analyse the metric on the moduli space of monopoles for the case $m = 2, k = 2$. Let us first consider the case $m = 1$.

Taub-NUT to flat metric: It is known [13, 3, 4] that the exact metric on the relative moduli space of fundamental $SU(3)$ monopoles is the Taub-NUT metric with positive mass parameter. If one of the monopoles becomes massless the Taub-NUT metric degenerates to flat metric on \mathbb{R}^4 .

In the notation of section 3.1 this is equivalent to $\lambda \rightarrow \infty$. Define $\nu = \lambda^{-1}$, so $\nu \rightarrow 0$ when $\lambda \rightarrow \infty$. In order for the action (20) to be well defined we must introduce a new parameter \tilde{t} and a new quaternionic coordinate \tilde{w} such that:

$$\nu \tilde{t} = t, \quad \tilde{w} = \nu w. \tag{66}$$

Then the action (20) becomes:

$$\begin{aligned} q &\rightarrow q e^{i\nu \tilde{t}}, \\ \tilde{w} &\rightarrow \tilde{w} + \nu \tilde{t}. \end{aligned} \tag{67}$$

In the limit $\nu \rightarrow 0$ the $U(1)$ action (67) is trivial and the metric on the quotient space X_0 is just the flat metric (12).

$\mathbf{SU}(\mathbf{m} + \mathbf{2}) \rightarrow \mathbf{SU}(\mathbf{m}) \times \mathbf{U}(\mathbf{1})^2$: Physically this situation corresponds to having two massive monopoles and $m - 1$ massless ones constituting a so-called massless cloud. We will see that in this case the Lee-Weinberg-Yi metric degenerates to the Taubian-Calabi metric of section 3.4.

When $m - 1$ monopoles become massless, the reduced mass matrix μ_{ab} drops in rank to rank = 1, which by (32) is equivalent to ν_{ab} having rank one and the translation vectors $v^{(a)}$ not being linearly independent. By analogy with (66) above we define new group parameters \tilde{t}_a and redefine quaternionic coordinates w_a as:

$$\nu_a{}^b \tilde{t}_b = t_a, \quad \tilde{w}_a = \nu_a{}^b w_b. \quad (68)$$

Then the action (27) becomes:

$$\begin{aligned} q_a &\rightarrow q_a e^{i\nu_a{}^b \tilde{t}_b}, \\ \tilde{w}_a &\rightarrow \tilde{w}_a + \nu_a{}^b \tilde{t}_b. \end{aligned} \quad (69)$$

When the rank of ν_{ab} is one there is only one independent coordinate \tilde{w}_a and the \mathbb{R}^m action (27) reduces to the \mathbb{R} action (69). All elements of μ_{ab} are equal and the Lee-Weinberg-Yi metric (31) on \mathbb{R}^{4m} degenerates to the Taubian-Calabi metric (45) on \mathbb{R}^{4m} . From section 3.4 the tri-holomorphic part of the full isometry group of the Taubian-Calabi metric that acts effectively is $U(m)/\mathbb{Z}_m$, which agrees with the result of [8] up to a discrete factor. In giving a physical interpretation to the degrees of freedom of the metric (45) Lee, Weinberg and Yi pointed out that the massless monopoles cannot be regarded as individual particles. Instead they form a so-called massless cloud that carries no net non-abelian charge and is characterised by one “size” parameter $R = \sum r_a$. This quantity is clearly invariant under the full isometry group $U(m) \times SU(2)$ since the metric on \mathcal{M} , and consequently $\sum q_a \bar{q}_a$, is preserved by both the $U(m)$ action and the $SU(2)$ action, but $q_a \bar{q}_a = |q_a i \bar{q}_a| = r_a$. So R is an invariant of the isometry group.

Let us focus on the simplest non-trivial case $m = 2$.

$\mathbf{SU}(4) \rightarrow \mathbf{SU}(2) \times \mathbf{U}(1)^2$: The Taubian-Calabi metric (45) on \mathbb{R}^8 is:

$$\begin{aligned} 4ds^2 &= \left(1 + \frac{1}{r_1}\right) d\mathbf{r}_1^2 + 2d\mathbf{r}_1 \cdot d\mathbf{r}_2 + \left(1 + \frac{1}{r_2}\right) d\mathbf{r}_2^2 \\ &\quad + \frac{1}{1 + r_1 + r_2} \left[r_1(1 + r_2)(d\tau_1 + \boldsymbol{\omega}(\mathbf{r}_1) \cdot d\mathbf{r}_1)^2 \right. \\ &\quad \left. - 2r_1 r_2 (d\tau_1 + \boldsymbol{\omega}(\mathbf{r}_1) \cdot d\mathbf{r}_1)(d\tau_2 + \boldsymbol{\omega}(\mathbf{r}_2) \cdot d\mathbf{r}_2) + r_2(1 + r_1)(d\tau_2 + \boldsymbol{\omega}(\mathbf{r}_2) \cdot d\mathbf{r}_2)^2 \right], \end{aligned} \quad (70)$$

where

$$\text{curl} \boldsymbol{\omega}_i = \text{grad} \left(\frac{1}{r_i} \right), \quad i = 1, 2.$$

The action of $U(2)$ is tri-holomorphic and preserves $\mathbf{r}_1 + \mathbf{r}_2$, it has 4-dimensional orbits. The left $SU(2)$ action preserves the length of any vector and the inner products but rotates the vectors around. Thus it rotates $\mathbf{r}_1 + \mathbf{r}_2$ but keeps $|\mathbf{r}_1 + \mathbf{r}_2|$ invariant. This makes the principle orbits 6-dimensional.

The metric (70) describes the moduli space of distinct centred fundamental $SU(4)$ monopoles in the limit when two of them become massless. So we are left with two fundamental monopoles charged with respect to the two $U(1)$ ’s and a “massless” cloud. Now there are 8 parameters on the moduli space: $\mathbf{r}_1, \tau_1, \mathbf{r}_2, \tau_2$. Four of them correspond to position and phase of the two massive monopoles relative to centre of mass coordinates; so there are four parameters left to

describe the cloud. Note that although we call the cloud “massless” it has non-zero moment of inertia, in fact it has infinite moment of inertia. It has zero mass in the sense of a point particle, but cannot be regarded as such since it has a finite size. Infinite inertia means that it would take an infinite amount of energy to rotate this smeared out cloud. A good analogy to this situation is provided by \mathbb{CP}^1 lumps which are rational maps from $\mathbb{C} \cup \{\infty\}$ onto \mathbb{CP}^1 . The kinetic energy metric fails to converge in some directions in moduli space because the lumps have infinite moment of inertia [20].

One may ask how the metric behaves if the cloud is large, i.e. if $|\mathbf{r}_1 + \mathbf{r}_2| \gg |\mathbf{r}_1 - \mathbf{r}_2|$, the left-hand side is equivalent to R , the cloud size parameter. Since swapping q_1 and q_2 is an isometry that preserves the HyperKähler structure, the submanifold $q_1 = q_2$ of the fixed points of this isometry, which implies $\mathbf{r}_1 = \mathbf{r}_2$ and $\tau_1 = \tau_2$, is totally geodesic. In fact, it is isomorphic to the Taub-NUT space. Therefore for large separations the cloud part of the metric, whose coordinates are $(\mathbf{r}_1 + \mathbf{r}_2)$ and $\frac{1}{2}(\tau_1 + \tau_2)$, behaves like the Taub-NUT metric. This is consistent with the fact that the cloud has four degrees of freedom – the position of its centre of mass and a phase. It has no rotational degrees of freedom, which is consistent with the claim that its moment of inertia is infinite.

$SU(\mathbf{m} + 2) \rightarrow SU(\mathbf{m} + 2 - \mathbf{k}) \times U(1)^{\mathbf{k}}$: We can now construct all the intermediate cases when the broken gauge group $SU(m + 2)$ contains one non-abelian factor. These can be viewed as generalisations of the Taubian-Calabi metrics (45) with the initial HyperKähler manifold now being

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H}^{k-1}.$$

Now $(m + 1 - k)$ monopoles become massless, $k = 2, \dots, m$, so rank of μ_{ab} and rank of ν_{ab} is equal to $k - 1$. Looking at the action (69) we can see that there are only $k - 1$ independent coordinates \tilde{w}_i , $i = 1, \dots, k - 1$. The potential function G_{ab} is the same as in (31) with μ_{ab} of the form:

$$\mu_{ab} = \left(\begin{array}{c|c} \mu'_{ab} & \cdots \\ \vdots & \ddots \end{array} \right)$$

where μ'_{ab} is the $(k - 1) \times (k - 1)$ reduced mass matrix for k fundamental monopoles with the rest of the entries all equal. It is not difficult to see that the isometry group for this manifold up to a discrete factor will be:

$$SU(2) \times U(m - (k - 2)) \times U(1)^{k-2}.$$

4.2 Infinitely massive monopoles

Here we will consider another interesting degeneration of the Lee-Weinberg-Yi metric which occurs when the translation matrix λ_{ab} drops in rank. Let λ_{ab} have rank one. Then the \mathbb{R}^m action (27)

$$\begin{aligned} q_a &\rightarrow q_a e^{it_a} \quad (\text{no sum over } a) \\ w_a &\rightarrow w_a + \lambda_a^b t_b \end{aligned} \tag{71}$$

reduces to the \mathbb{R}^m action (55), and there is effectively one w_a coordinate. The initial manifold \mathcal{M} is now $\mathbb{H}^m \times \mathbb{H}$ and the setup is equivalent to the setup for cyclic ALF spaces in section 3.6. Physically this represents situation where moduli space coordinates of all but one monopoles are fixed (all except one monopoles are infinitely heavy) and one monopole (described by the one

w_a coordinate) moves in their background. We get the multi-centre metric with ALF boundary conditions (56).

If the rank of λ_{ab} is $1 < k < m$, we get a $4k$ -dimensional generalisation of cyclic ALF spaces.

5 Geodesics and Integrability

The slow *classical* motion of monopoles corresponds to geodesics on the moduli space [21]. It is of interest to know whether there exist any closed or bound⁴ geodesics which would describe (classical) bound states of monopoles. Note that here we are not talking about the zero-energy threshold bound states predicted by Sen's conjecture, but about true bound states with positive energy.

If the topology of the moduli space is complicated one may invoke a general result of Benci and Giannoni [22] for open manifolds to establish existence of closed geodesics. However if the manifold is topologically trivial, such arguments give no information.

For topologically trivial manifolds such as Lee-Weinberg-Yi and Taubian-Calabi spaces one may use the following criteria. If there exists an everywhere distance increasing vector field \mathbf{V} then there are no closed or bounded geodesics on this manifold. Distance increasing condition means that the Lie derivative of the metric along \mathbf{V} satisfies:

$$\mathcal{L}_{\mathbf{V}}g(\mathbf{X}, \mathbf{Y}) > 0, \quad (72)$$

for all $4m$ -vectors \mathbf{X}, \mathbf{Y} , or equivalently

$$\mathbf{V}_{(a;b)}\mathbf{X}^a\mathbf{Y}^b > 0.$$

Along a geodesic with a tangent vector \mathbf{L} one therefore has:

$$\frac{d}{dt}g(\mathbf{V}, \mathbf{L}) = \mathcal{L}_{\mathbf{V}}g(\mathbf{L}, \mathbf{L}) > 0. \quad (73)$$

Now if this is a bound or closed geodesic one may average over a time period T . The left-hand side of (73) tends to zero as $T \rightarrow \infty$ while the right-hand side tends to some positive constant, which is a contradiction.

The existence of a distance increasing vector field can be easily demonstrated on spaces obtained by HyperKähler quotient restricting to zero-set of the moment map⁵. From our examples in section 3 these are Lee-Weinberg-Yi and Taubian-Calabi manifolds. The vector field \mathbf{V} is induced on X_0 from the following \mathbb{R}^+ action on $\mathcal{M} = \mathbb{H}^m \times \mathbb{H}^p$:

$$\begin{aligned} q_a &\rightarrow \alpha^{1/2} q_a \\ w_i &\rightarrow \alpha w_i, \quad \alpha > 0 \end{aligned} \quad (74)$$

$a = 1, \dots, m$ and $i = 1, \dots, p$. This \mathbb{R}^+ action leaves invariant the level sets $\mu^{-1}(0)$ and commutes with the \mathbb{R}^m , T^m and $SU(2)$ actions. It therefore descends to give a well-defined \mathbb{R}^+ action on $\mu^{-1}(0)/\mathbb{R}^m$ which stabilises the point $q_a = 0$ corresponding for Lee-Weinberg-Yi metric to the spherically symmetric monopole [4]. The action (74) is clearly distance increasing

⁴By bound we mean confined to a compact set for all times.

⁵It will be clear shortly why this argument does not apply to spaces where one cannot without loss of generality consider $\mu^{-1}(0)$.

on \mathcal{M} , so its restriction to $\mu^{-1}(0)/\mathbb{R}^m$ is also distance increasing. Note that the argument just given is a more geometric version of the generalised Virial Theorem given earlier in [5].

We conclude this section by making a remark about the integrability of the geodesic flow. Consider the Lagrangian for the configuration described by the moduli space metric (13) (see [7]). If one eliminates the conserved charges

$$Q^a = G^{ab} \left(\frac{d\tau_a}{dt} + \omega_{ac} \cdot \frac{d\mathbf{r}_c}{dt} \right),$$

one obtains an effective Lagrangian on $\mathbb{R}^{3m} = X_\zeta/T^m$:

$$G_{ab} \frac{d\mathbf{r}_a}{dt} \cdot \frac{d\mathbf{r}_b}{dt} - G_{ab} Q^a Q^b + Q^a \omega_{ac} \cdot \frac{d\mathbf{r}_c}{dt}. \quad (75)$$

This many-body Lagrangian (75) may not look very tractable when considered on \mathbb{R}^{3m} but in some cases it admits “hidden” symmetries which, although not apparent in $3m$ dimensions, are clearly present on the $4m$ -dimensional manifold. A simple example of the phenomenon occurs in the Eguchi-Hanson manifold. On \mathbb{R}^3 there is only one manifest symmetry corresponding to rotation about the axis joining the two centres. However, the geodesic motion is completely integrable [23]. This happens because of the large isometry group, $U(2)$, which acts on the three-dimensional orbits. In fact the motion on electrically neutral geodesics, i.e. those with $Q = 0$, in the ALE and ALF spaces associated to the cyclic group of order k is the same as that of a light planet moving in the Newtonian gravitational field of k fixed gravitating centres. The case $k = 1$ corresponds to the Kepler problem and is clearly integrable, The case $k = 2$ corresponds to the Euler problem and is also integrable. According to [24] the case when there is a plane containing the centres and the forces are attractive and the motion of a planet with positive energy is confined to that plane, then there are no analytic constants of the motion other than the energy if $k > 2$. Since this case is a special case of the general motion, it strongly indicates that for $k > 2$ the geodesic flow on the cyclic ALE and ALF spaces is not integrable. One might be able to use a similar argument in some other cases.

However one might expect to encounter hidden symmetries in the case of the Calabi and the Taubian-Calabi metrics. As we have seen above neither the Calabi nor the Taubian-Calabi metrics look very symmetric when written out in terms of the cartesian coordinates but in fact they admit a large group of isometries whose principal orbits are of co-dimension one or two respectively. Another interesting question along the same lines is whether there are cases in which the geodesic flow admits Lagrange-Laplace-Runge-Lenz vectors as it does in the Taub-NUT case. We defer further considerations of these questions for a future publication.

6 Concluding Comments

In this paper we have, using the HyperKähler quotient, presented a rather simple and elegant way to construct and analyse some known and some new HyperKähler metrics. All our examples turned out to possess a tri-holomorphic torus action which considerably simplified the algebra. From the few applications that we have discussed, it is clear that this approach gives explicit answers to many interesting questions about the global properties of these manifolds. In many cases such properties are not immediately apparent from the local form of the metric.

There are a number of open problems that can be explored using the HyperKähler quotient. In the future we hope to extend the method introduced in this paper further to discuss the

differential forms and the spectrum of the Hodge-de Rahm Laplacian, as well as the physics of so-called massless monopoles. It would also be interesting to look at the singular metrics and understand the type of singularities that arise and how they may be resolved.

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